Towards the Asymptotic Sum Capacity of the MIMO Cellular Two-Way Relay Channel

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Abstract—In this paper, we consider the transceiver and relay design for the multiple-input multiple-output (MIMO) cellular two-way relay channel (cTWRC), where a multi-antenna base station (BS) exchanges information with multiple multi-antenna mobile stations via a multi-antenna relay station (RS). We propose a novel two-way relaying scheme to approach the sum capacity of the MIMO cTWRC. A key contribution of this work is a new non-linear lattice-based precoding technique to pre-compensate the inter-stream interference, so as to achieve efficient interference-free lattice decoding at the relay. We derive sufficient conditions for the proposed scheme to asymptotically achieve the sum capacity of the MIMO cTWRC in the high signal-to-noise ratio (SNR) regime. To fully exploit the potential of the proposed scheme, we also investigate the optimal power allocation at the BS and the RS to maximize the weighted sum-rate of the MIMO cTWRC in the large SNR regime. It is shown that the problem can be formulated as a monotonic program, and a polyblock outer approximation algorithm is developed to find the globally optimal solution with guaranteed convergence. We demonstrate by numerical results that the proposed scheme significantly outperforms the existing schemes and closely approaches the sum capacity of the MIMO cTWRC in the high SNR regime.

Index Terms—Cellular two-way relay channel, nested lattice coding, lattice precoding, monotonic optimization.

I. INTRODUCTION

Two-way communications can be dated back to Shannon [1] and has been rediscovered as an efficient method to mitigate the loss of spectral efficiency in conventional half-duplex one-way relaying [2]–[9]. Tremendous progress has been made for efficient communications over the two-way relay channel (TWRC), in which two users want to exchange information via the help of a single relay. The main idea, termed physical-layer network coding (PNC), is to allow the two users to communicate with the relay simultaneously, and to allow each user to decode the message from the other user by exploiting the knowledge of the self-message. It was shown in [6] that PNC with nested lattice coding can achieve the cut-set outer bound of the single-input single-output Gaussian TWRC within 1/2 bit. Later, the authors in [7]–[9] studied the multiple-input multiple-output (MIMO) TWRC, where both the users and the relay are equipped with multiple antennas. It was shown that near-capacity performance can be achieved in the MIMO TWRC by using nested lattice coding aided PNC.

PNC design for more sophisticated relay networks has recently attracted much research interest. In this regard, the authors in [10] generalized the TWRC model to the multiway relay channel (mRC), in which a relay simultaneously serves multiple clusters of users, and each user in a cluster wants to multicast its message to all the other users in the same cluster. Several special cases of the mRC have been studied in the literature. For example, the authors in [11]–[14] considered the multi-pair MIMO TWRC, which is a special case of the mRC with each cluster consisting of two users; also, the Y channel proposed in [15] is a special case of the mRC with only one cluster. Various relaying protocols, including amplify-and-forward (AF), decode-and-forward (DF), compress-and-forward, and their mixtures, were investigated for these relay networks.

In this paper, we investigate efficient PNC design for another important two-way relaying model, termed the cellular TWRC (cTWRC), where multiple users in a cellular network want to exchange information with a multiple-antenna base station (BS) via the help of a multiple-antenna relay station (RS). It has been shown that two-way relaying has the potential to increase network throughput and extend the coverage, and is a promising technique for future cellular systems [16]. For this reason, a MIMO cTWRC model was previously studied in [17]–[22]. In this model, the information exchange is realized using a two-phase protocol: In the first phase both the BS and the users transmit signals to the BS; in the second phase, the relay broadcasts signals to the BS and the users. In [17], linear precoding was applied at the BS to align the signals impinging upon the relay in such a way that each signal stream of the BS is aligned to the direction of the user’s signal stream to be exchanged with. In [18] and [19], linear precoding was applied at both the BS and the relay, and iterative algorithms were proposed to optimize the corresponding precoders based on various design criteria, such as sum-rate maximization or max-min signal-to-interference-plus-noise ratio (SINR). However, all these approaches were based on AF relaying which generally suffers from the noise propagation, as well as from the power inefficiency caused by transmitting analogue (instead of algebraic) superposition of the BS and user signals at the relay.

To avoid the aforementioned disadvantages, the authors in
 proped a DF-based relaying scheme for the MIMO cTWRC involving linear precoding and nested lattice coding at the BS and users, and dirty-paper coding at the RS. It was shown that the achievable sum-rate of this scheme is much higher than the AF based schemes in [17]–[19], and this scheme can achieve the cut-set outer bound of the MIMO cTWRC if only the second phase is concerned. However, the scheme in [21] can perform far away from the capacity of the MIMO cTWRC, especially for a relatively large MIMO setup. This performance gap is largely due to the following fact: In the first phase, linear precoding can be applied only at the BS (as the users cannot cooperate), whereas the BS precoder alone fails to provide enough freedom to align the signals efficiently for interference-free PNC decoding at the relay.

In this paper, we propose a novel nested-lattice-coding aided PNC scheme to approach the sum capacity of the MIMO cTWRC. Compared to [21], a major difference and contribution of this work is a new non-linear precoding technique, called lattice precoding, employed in the first-phase system design. We show that, together with linear precoding, nested lattice coding, and successive interference cancellation (SIC), the proposed lattice precoder at the BS efficiently precompensates for the inter-stream interference (ISI) seen at the relay, such that interference-free lattice decoding can be performed at the relay. We derive the achievable rates of the proposed scheme, and establish sufficient conditions for the proposed scheme to asymptotically achieve the sum capacity of the MIMO cTWRC in the high SNR regime. Furthermore, we formulate a weighted sum-rate maximization problem for the proposed scheme to optimize the power allocation of the nodes in the network and show that this non-convex problem is solvable using monotonic programming (MP) [23]. An efficient polyblock outer approximation algorithm is developed to find the optimal power allocation. Numerical results demonstrate that the proposed scheme significantly outperforms its existing alternatives and closely approaches the capacity of the MIMO cTWRC at high SNR.

The rest of this paper is organized as follows. Section II describes the MIMO cTWRC model. Section III presents the proposed encoding and decoding scheme, while Section IV analyzes the achievable sum-rate of the proposed scheme. Section V investigates the optimal power allocation problem. Section VI discusses the extension of the proposed scheme to MIMO cTWRCs with general antenna setups. The proposed scheme is tested and compared with the existing schemes in Section VII, followed by the conclusions in Section VIII.

Notation: The following notation is used throughout this paper. Boldface letters denote vectors or matrices. $(\cdot)^T$ denotes transpose. The $(i,j)$-th element of a matrix $A$ is denoted by $a(i,j)$. $\mathbb{R}^{K \times M}$ and $\mathbb{C}^{K \times M}$ denote the $K$-by-$M$ dimensional real and complex space, respectively. $\| \cdot \|_F$ denotes the Frobenius norm. tr(\cdot) denotes the trace operation; $\text{diag}(a_1, \ldots, a_N)$ denotes a diagonal matrix with diagonal elements $(a_1, \ldots, a_N)$, and $(A)_{\text{diag}}$ denotes the diagonal matrix specified by the diagonal of matrix $A$; $I_K$ denotes a $K \times K$ identity matrix; $e_i$ denotes a unit vector with the only non-zero element in the $i$-th entry; $[x]_+ \triangleq \max(x, 0)$.

![Fig. 1. A MIMO cTWRC with $K$ mobile stations.](image)

II. SYSTEM MODEL

We consider a MIMO cTWRC where a BS communicates with $K$ mobile stations (MSs) via a single relay station, as shown in Fig. 1. There is no direct link between the BS and the MSs. The BS, the RS, and the $k$-th MS are equipped with $N_B, N_R$, and $N_{M,k}$ antennas, respectively. We consider quasi-static flat-fading channels where the channel coefficients keep unchanged in the duration of a transmission frame, denoted by $T$. All the nodes are half-duplex and the bidirectional transmission takes place in two phases. For presentation clarity, we consider single-antenna MSs, i.e., $N_{M,k} = 1$, for $k = 1, \ldots, K$, and assume $N_B = N_R = K$. The extension to a general antenna setup will be discussed in Section VI. Each MS exchanges one data stream with the BS, and there are $2K$ data streams in total. The channel matrix from the BS to the MS is denoted by $H_{BR} \in \mathbb{C}^{K \times K}$, and the channel vector from the $k$-th MS to the RS by $h_{k,R} \in \mathbb{C}^{K \times 1}$. $H_{RB} \in \mathbb{C}^{K \times K}$ and $h_{R,k} \in \mathbb{C}^{1 \times K}$ are the corresponding channel matrix/vector for the reverse links. Following the convention (e.g., in [18] and [21]), we assume that all the nodes have global channel state information of all links.

The transmission protocol is described as follows. In the first phase, the BS and all the $K$ MSs transmit to the RS simultaneously. Let $X_B \in \mathbb{C}^{K \times T}$ and $x_{M,k} \in \mathbb{C}^{1 \times T}$ denote the transmit signal at the BS and the $k$-th MS, respectively. The received signal at the RS is given by

$$Y_R = H_{BR}X_B + H_{MR}X_M + \Psi_R,$$

where $H_{MR} := [h_{1,R}, \ldots, h_{K,R}] \in \mathbb{C}^{K \times K}$, $X_M := [x_{M,1}^T, \ldots, x_{M,K}^T]^T \in \mathbb{C}^{K \times T}$, and $\Psi_R \in \mathbb{C}^{K \times T}$ denotes the additive white Gaussian noise (AWGN) at the RS. It is assumed that each element in $\Psi_R$ is independent and identically distributed (i.i.d.) with zero mean and a variance of $\sigma^2$. The maximum transmit powers at the BS and the $k$-th MS are $P_B$ and $P_{M,k}$, respectively, i.e., $\frac{1}{T}\text{tr}(X_BX_B^H) \leq P_B$, and $\frac{1}{T}\|x_{M,k}\|_F^2 \leq P_{M,k}$.

Upon receiving $Y_R$, the transmit signal at the RS is regenrated as $X_R = g_R(Y_R) \in \mathbb{C}^{K \times T}$, where $g_R(\cdot)$ denotes the RS decoding and re-encoding function. The transmit power of the RS is constrained as $\frac{1}{T}\text{tr}(X_RX_R^H) \leq P_R$, where $P_R$ is the power budget at the relay.
In the second phase, the RS broadcasts $X_R$ to the BS and the $K$ MSs. The received signals at the BS and the $k$-th MS are given by

$$Y_B = H_{RB}X_R + \Psi_B, \quad y_{M,k} = h_{R,k}X_R + \psi_{M,k},$$

where $\Psi_B$ and $\psi_{M,k}$ are the AWGN at the BS and the $k$-th MS, respectively. With $Y_M = [y_{M,1}^T, \ldots, y_{M,K}^T]^T$, $H_{RM} = [h_{R,1}^T, \ldots, h_{R,K}^T]^T$, and $\Psi_M = [\psi_{M,1}^T, \ldots, \psi_{M,K}^T]^T$, we have

$$Y_M = H_{RM}X_R + \Psi_M.$$  

With the knowledge of the self-message $X_B$, the BS estimates all the MS messages from the received signal $Y_B$. Meanwhile, for each $k \in \{1, \ldots, K\}$, the $k$-th MS decodes the intended private message of the BS from $y_{M,k}$ with the knowledge of $x_{M,k}$.

For the MIMO cTWRC, let $R_{B,k}$ be the transmission rate from the BS to the $k$-th MS, and $R_{M,k}$ be the transmission rate from the $k$-th MS to the BS. A rate tuple $(R_{B,1}, \ldots, R_{B,K}, R_{M,1}, \ldots, R_{M,K})$ is said to be achievable if there exist transmit encoding functions, MIMO processing functions, and receive decoding functions at the BS, RS, and MSs such that the decoding error probabilities tend to zero as the codeword length $T \to \infty$. From the cut-set theorem, two sum-rate outer bounds of the MIMO cTWRC are given by [21]

$$\sum_{k=1}^K R_{B,k} \leq \min \left\{ \frac{1}{2} \log |I_K + H_{BR}Q_{BR}H_{BR}^H|, \quad \frac{1}{2} \log |I_K + H_{RM}Q_{RM}H_{RM}^H| \right\}, \quad (5a)$$

$$\sum_{k=1}^K R_{M,k} \leq \min \left\{ \frac{1}{2} \log |I_K + H_{MR}Q_{MR}H_{MR}^H|, \quad \frac{1}{2} \log |I_K + H_{RB}Q_{RB}H_{RB}^H| \right\}, \quad (5b)$$

where $Q_S = \frac{1}{T}E(X_SX_S^H), S \in \{B, R, M\}$, are the corresponding signaling covariance matrices. These bounds will be used as a benchmark of the system design for the MIMO cTWRC.

### III. Proposed Two-way Relaying Scheme

In this section, we propose a novel two-phase two-way relaying scheme to approach the sum-rate capacity of the MIMO cTWRC. The key novelty of our scheme, compared to [21], is that lattice precoding and random dithering are employed in the first phase to pre-compensate for the inter-stream interference. Building on this theme, encoding and decoding operations at the BS, the RS, and the MSs are carefully designed to enable efficient interference-free PNC decoding at the relay, even with the restriction of non-cooperation among MSs.

#### A. Channel Triangularization

To start with, we describe a linear precoding technique to triangularize the channel matrices involved in the two transmission phases, following the approach in [21].

Consider the first phase. Let the QR decomposition of $H_{MR}$ be

$$H_{MR} = Q_{MR}R_{MR},$$

where $Q_{MR}$ is a unitary matrix and $R_{MR}$ is an upper-triangular matrix. Further let the RQ decomposition of $Q_{MR}^HH_{BR}$ be

$$Q_{MR}^HH_{BR} = R_{BR}Q_{BR},$$

where $Q_{BR}$ is a unitary matrix and $R_{BR}$ is an upper-triangular matrix. By multiplying $Q_{MR}^H$ to the BS received signal $Y_R$, we obtain

$$\tilde{Y}_R = Q_{MR}^HY_R = R_{BR}\tilde{X}_B + R_{MR}X_M + \tilde{\Psi}_R,$$

where $\tilde{X}_B = Q_{BR}X_B$, and $\tilde{\Psi}_R = Q_{MR}\Psi_R$.

Let $s_{B,k} \in \mathbb{C}^{1 \times T}$ be the coded vector of the BS transmitted to the $k$-th MS. The transmit signal of the BS is linearly precoded as $X_B = Q_{BR}^HS_B$, where $S_B = [s_{B,1}^T, \ldots, s_{B,K}^T]^T \in \mathbb{C}^{K \times T}$ is the codeword matrix. The transmit signal of the $k$-th MS is directly generated as $x_{M,k} = s_{M,k}$, where $s_{M,k} \in \mathbb{C}^{1 \times T}$ denotes the coded vector of the $k$-th MS. With such a linear precoding, the BS obtains

$$\tilde{Y}_R = R_{BR}S_B + R_{MR}S_M + \tilde{\Psi}_R,$$

where $S_M = [s_{M,1}^T, \ldots, s_{M,K}^T] \in \mathbb{C}^{K \times T}$. Correspondingly, the power constraints at the BS and the $k$-th MS can be equivalently written as $\frac{1}{T}\text{tr}(S_BS_B^H) \leq P_B$, and $\frac{1}{T}\|s_{M,k}\|^2 \leq P_{M,k}, k = 1, \ldots, K$.

The channels from the RS to the BS and MSs can be triangularized in a similar way. Let $\Phi$ be an arbitrary permutation matrix. We will see in Section III-E that $\Phi$ specifies the DPC re-encoding order at the relay. Let the LQ decomposition of the re-ordered channel matrix $\Phi H_{RM}$ be

$$\Phi H_{RM} = L_{RM}Q_{RM},$$

where $Q_{RM}$ is unitary and $L_{RM}$ is lower-triangular.

The transmit signal of the RS is precoded as

$$X_R = Q_{RM}^H X_{R,DPC},$$

where $X_{R,DPC} \in \mathbb{C}^{K \times T}$ is the DPC codeword matrix to be elaborated in Subsection E. As $Q_{RM}$ is unitary, the power constraint of the relay can be equivalently written as $\frac{1}{T}\text{tr}(X_{R,DPC}X_{R,DPC}^H) \leq P_R$.

The permuted received signal $\tilde{Y}_M = \Phi Y_M$ at all $K$ MSs can be expressed as

$$\tilde{Y}_M = \Phi H_{RM}X_R + \Phi \Psi_M = L_{RM}X_{R,DPC} + \tilde{\Psi}_M,$$

where $\tilde{\Psi}_M = \Phi \Psi_M$ is still an AWGN matrix. Note that there is only one non-zero entry in each row and column of the permutation matrix $\Phi$. For a given $\Phi$, $\Phi Y_M$ gives a matrix with reordered rows of $Y_M$. This operation doesn’t involve any joint processing of the signals from different MSs. Therefore, the permutation operation $\tilde{Y}_M = \Phi Y_M$ is
possible, even when the MSs cannot cooperate. Based on the signal model in (12), we will show in Section III-F that the achievable rates of the RS-MS link depend on the diagonal of \( L_{RM} \). On the other hand, it follows from (10) that the lower-triangular matrix \( L_{RM} \) varies with the choice of the permutation matrix \( \Phi \). Hence, the sum-rate performance of the proposed scheme can be improved by searching the optimal permutation matrix \( \Phi \).

Now consider the received signal at the BS. Let the QL decomposition of \( H_{RB} Q_{RM}^H \) be

\[
H_{RB} Q_{RM}^H = Q_{RB} L_{RB}, \tag{13}
\]

where \( Q_{RB} \) is unitary and \( L_{RB} \) is lower-triangular. By multiplying \( Q_{RB} \) to the received signal \( Y_B \) in (2), the BS obtains

\[
\hat{Y}_B = Q_{RB}^H Y_B = L_{RB} X_{R,DPC} + \tilde{\Psi}_B, \tag{14}
\]

where \( \tilde{\Psi}_B = Q_{RB}^H \Psi_B \). In the above channel triangularization, only unitary transforms are involved, implying that the new signal model in (9), (12) and (14) has the same capacity as the original MIMO cTWRC. Therefore, we henceforth focus on the signaling design for the equivalent MIMO cTWRC given by (9), (12) and (14).

### B. Partitions of Upper-Triangular Matrices \( R_{BR} \) and \( R_{MR} \)

In the following, we describe the main ideas behind our novel lattice precoding and decoding scheme, based on channel models given in [9], [12] and (14). We first consider the phase-1 channel model in (9). From (9), we can see that \( R_{BR} \) and \( R_{MR} \) are both upper-triangular matrices. Hence, ISI generally exists in relay decoding. Our objective is to cancel the ISI by combining lattice precoding at the BS and successive interference cancellation at the relay. To this end, we rewrite \( R_{BR} \) and \( R_{MR} \) as \( R_{BR} = R_{BR}' + (R_{BR} - R_{BR}') \) and \( R_{MR} = (R_{MR})_{\text{diag}} + (R_{MR} - (R_{MR})_{\text{diag}}) \), with the corresponding signal model (9) re-written as

\[
\hat{Y}_R = \left( R_{BR}' S_B + (R_{MR})_{\text{diag}} S_M \right) \text{ signal to be decoded at the RS} \nonumber
\]

\[
+ \left( (R_{BR} - R_{BR}') S_B + (R_{MR} - (R_{MR})_{\text{diag}}) S_M \right) \text{ interference to be cancelled at the RS} \nonumber
\]

\[
(15)
\]

where \( R_{BR}' \) is an upper-triangular matrix to be determined shortly. The first term in the right-hand side (RHS) of (15), \( \hat{S} \triangleq \left( R_{BR}' S_B + (R_{MR})_{\text{diag}} S_M \right) \in \mathbb{C}^{K \times T} \), represents the signal to be decoded at the RS. The second term in the RHS of (15), \( \hat{W} \triangleq \left( (R_{BR} - R_{BR}') S_B + (R_{MR} - (R_{MR})_{\text{diag}}) S_M \right) \in \mathbb{C}^{K \times T} \), denotes the residual inter-stream interference. We need to properly choose \( R_{BR}' \) such that this interference term can be successively cancelled at the RS with decoding ordered from the \( K \)-th spatial stream to the first stream. First, to ensure that the second term \( \hat{W} \) in (15) only contains inter-stream interference, \( (R_{BR} - R_{BR}') \) is required to be strictly upper-triangular, i.e., the diagonal of \( R_{BR}' \) is chosen the same as that of \( R_{BR} \). Let \( \hat{s}^{(k)} \in \mathbb{C}^{1 \times T} \) and \( \tilde{w}^{(k)} \in \mathbb{C}^{1 \times T} \) be the \( k \)-th row of \( \hat{S} \) and \( \hat{W} \), respectively. In decoding the \( k \)-th network-coded message \( \hat{s}^{(k)} \), the relay already knows the \((k + 1)\)-th to \( K\)-th network-coded messages \( \hat{s}^{(k+1)}, \ldots, \hat{s}^{(K)} \), since the decoding is ordered from the \( K \)-th spatial stream to the first stream. If the \( k \)-th inter-stream interference term \( \tilde{w}^{(k)} \) can be expressed as a linear combination of the \((k + 1)\)-th to \( K\)-th network-coded messages \( \hat{s}^{(k+1)}, \ldots, \hat{s}^{(K)} \), then the relay is able to completely remove \( \tilde{w}^{(k)} \) from the received signal. This is equivalent to say that there exists a strictly upper-triangular matrix \( U_R \in \mathbb{C}^{K \times K} \).

\[
U_R = \begin{pmatrix}
0 & u_R(1,2) & u_R(1,3) & \cdots & u_R(1,K) \\
0 & 0 & u_R(2,3) & \cdots & u_R(2,K) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \tag{16}
\]

which satisfies

\[
U_R \hat{S} = \hat{W}, \tag{17}
\]

for arbitrary \( S_B \) and \( S_M \). The above condition can be further written as

\[
U_R R_{BR}' = R_{BR} - R_{BR}', \tag{18}
\]

\[
U_R (R_{MR})_{\text{diag}} = R_{MR} - (R_{MR})_{\text{diag}}. \tag{19}
\]

From (18), \( R_{BR}' \) can be expressed as

\[
R_{BR}' = (I_K + U_R)^{-1} R_{BR}. \tag{20}
\]

From (19), we have

\[
U_R = (R_{MR} - (R_{MR})_{\text{diag}})(R_{MR})_{\text{diag}}^{-1}. \tag{21}
\]

Substituting (21) into (20), we further obtain

\[
R_{BR}' = (I_K + (R_{MR} - (R_{MR})_{\text{diag}})(R_{MR})_{\text{diag}}^{-1})^{-1} R_{BR} = (R_{MR})_{\text{diag}}^{-1} R_{BR}. \tag{22}
\]

With such choices of \( U_R \) and \( R_{BR}' \), the relay is then able to remove the inter-stream interference \( \hat{W} \) successively with decoding ordered from the \( K \)-th spatial stream \( \hat{s}^{(K)} \) to the first stream \( \hat{s}^{(1)} \).

To see it, let \( \hat{y}_R^{(k)} \) and \( \hat{\psi}_R^{(k)} \) be the \( k \)-th row of \( \hat{Y}_R \) and \( \hat{\Psi}_R \), respectively. Then, from (15), the \( k \)-th subchannel can be expressed as

\[
\hat{y}_R^{(k)} = \hat{s}^{(k)} + \tilde{w}^{(k)} + \hat{\psi}_R^{(k)}. \tag{23}
\]

From (16) and (17), the interference term \( \tilde{w}^{(k)} \) can be expressed as

\[
\tilde{w}^{(k)} = \sum_{n=k+1}^{K} u_R(k,n) \hat{s}^{(n)}, \tag{24}
\]

which implies that the interference term \( \tilde{w}^{(k)} \) is a linear combination of the signals \( \{\hat{s}^{(k+1)}, \ldots, \hat{s}^{(K)}\} \). Note that the decoding is ordered from the \( K \)-th stream to the first stream. When decoding the \( k \)-th spatial stream \( \hat{s}^{(k)} \), \( \{\hat{s}^{(k+1)}, \ldots, \hat{s}^{(K)}\} \) are already decoded and are known to the
relay. As a result, the residue ISI \( \tilde{w}^{(k)} \) can be completely removed from the received signal \( \tilde{y}^{(k)} \):

\[
\tilde{z}^{(k)}_R = \tilde{y}^{(k)}_R - \tilde{w}^{(k)} = \tilde{s}^{(k)} + \tilde{\psi}^{(k)}_R.
\]

(25)

After successive interference cancellation, the \( k \)-th subchannel scaled by the factor \( \frac{1}{r_{BR(k,k)}} \), can be expressed as

\[
y_{R,k} = \frac{\tilde{z}^{(k)}_R}{r_{BR(k,k)}} = \frac{\tilde{s}^{(k)}_R + \tilde{\psi}^{(k)}_R}{r_{BR(k,k)}},
\]

(26)

where

\[
\alpha_k = \frac{r_{MR}(k,k)}{r_{BR(k,k)}},
\]

(27a)

\[
v_k = \sum_{j=\max(0,k-1)}^{K-1} \frac{r_{BR(k,j)}}{r_{BR(k,k)}} s_{B,j},
\]

(27b)

\[
\tilde{\psi}^{(k)}_R = \frac{\tilde{\psi}^{(k)}_R}{r_{BR(k,k)}}.
\]

(27c)

Note that the decoded signal of the \( k \)-th stream is \( s_{B,k} + \alpha_k s_{M,k} + v_k \). Here, \( v_k \) in (27b) is a weighted sum of the last \((K-k)\) rows of \( S_B \). Let the encoding order of the BS be also from the \( K \)-th spatial stream to the first stream. Then, when encoding the \( k \)-th spatial stream at the BS, \( v_k \) is known to BS; thus it can be pre-cancelled using lattice precoding at the BS, as will be elaborated in the sequel.

C. Phase 1: Encoding at the BS and MSs

In the following, we describe the encoding and decoding operations involved in the proposed scheme. We start with the encoding operations at the BS and MSs.

Let \( \mathcal{W}_{M,k} = \{1,2,\ldots,2^{TR_{M,k}}\} \) be the message set for the data stream at the \( k \)-th MS, and \( w_{M,k} \in \mathcal{W}_{M,k} \) be the corresponding message. The \((1 \times T)\)-dimensional coded vector for the spatial data stream at the \( k \)-th MS is denoted as \( s_{M,k} = f_{M,k}(w_{M,k}) \), where \( f_{M,k}(\cdot) \) is the encoding function to be specified in the following. Similarly, the message of the \( k \)-th spatial stream transmitted to the \( k \)-th MS at the BS is denoted by \( w_{B,k} \in \mathcal{W}_{B,k} \), where \( \mathcal{W}_{B,k} = \{1,2,\ldots,2^{TR_{B,k}}\} \) is the message set. The corresponding \((1 \times T)\)-dimensional encoded vector is denoted by \( s_{B,k} = f_{B,k}(w_{B,k}) \).

Nested lattice coding is applied to each message pair \((w_{B,k},w_{M,k})\). An \( n \)-dimensional lattice \( \Lambda \) is a subgroup of \( \mathbb{R}^n \) under normal vector addition. A lattice \( \Lambda \) is nested in the lattice \( \Lambda_1 \) if \( \Lambda \subseteq \Lambda_1 \). The main idea of nested lattice codes is to use the coarse lattice \( \Lambda \) as a shaping region and the lattice points from the fine lattice \( \Lambda_1 \) within the Voronoi region of the coarse lattice \( \Lambda \) as the codewords. The details of nested lattice codes can be found in [6], [24]–[26].

The encoding functions \( \{f_{B,k}(\cdot)\}_{k=1}^{K} \) and \( \{f_{M,k}(\cdot)\}_{k=1}^{K} \) are described as follows. The encoding of the BS follows the order from the \( K \)-th stream to the first stream sequentially. Without loss of generality, we assume \( R_{B,k} \leq R_{M,k} \). For the \( k \)-th subchannel in (26), we construct a nested lattice chain \( \Lambda_{B,k}, \Lambda_{M,k} \) and \( \Lambda_{C,k} \) satisfying \( \Lambda_{M,k} \subseteq \Lambda_{B,k} \subseteq \Lambda_{C,k} \). Here, \( \Lambda_{B,k} \) and \( \Lambda_{M,k} \) are simultaneously Rogers-good and Poltyrev-good while \( \Lambda_{C,k} \) is Poltyrev-good [6]. Let \( c_{X,k} \in \mathcal{C}_{X,k} \) denote the codeword mapped from the message \( w_{X,k} \), \( X \in \{B,M\} \), where \( \mathcal{C}_{X,k} \) is the nested lattice code defined by \( \Lambda_{X,k} \) and \( \Lambda_{C,k} \). The coding rate of the nested lattice code \( \mathcal{C}_{X,k} \) can approach any \( \xi_X > 0 \) as \( T \rightarrow \infty \) [6], i.e.

\[
R_{X,k} = \frac{1}{T} \log \left( \frac{\Vol(\Lambda_{X,k})}{\Vol(\Lambda_{C,k})} \right) = \xi_X + o_T(1),
\]

(28)

where \( \Vol(\Lambda) \) is the volume of the Voronoi region of a lattice \( \Lambda \), and \( o_T(1) \rightarrow 0 \) as \( T \rightarrow \infty \).

For the \( k \)-th subchannel in (26), \( \Lambda_{B,k} \) and \( \Lambda_{M,k} \) are chosen to meet

\[
\left( \frac{\Vol(\Lambda_{M,k})}{\Vol(\Lambda_{B,k})} \right)^{\frac{1}{2}} = \left( \frac{r_{MR}(k,k)^2 P_{M,k}}{r_{BR(k,k)}^2 P_{B,k}} \right)^{\frac{1}{2}} + o_T(1),
\]

(29)

where \( P_{B,k} \) denotes the average power of the \( k \)-th spatial stream at the BS. Then, the relation between \( R_{B,k} \) and \( R_{M,k} \) can be written as [6]

\[
R_{B,k} = R_{M,k} + \frac{1}{2} \log \left( \frac{r_{BR(k,k)}^2 P_{B,k}}{r_{MR}(k,k)^2 P_{M,k}} \right) + o_T(1).
\]

(30)

The encoding at the BS is as follows. Let \( \delta_{X,k} \) be a random dithering vector that is uniformly distributed over the Voronoi region of \( \Lambda_{X,k} \), \( X \in \{B,M\} \). With random dithering, the \( k \)-th transmit signal \( s_{B,k} \) at the BS is constructed as

\[
s_{B,k} = (c_{B,k} - v_k - d_{B,k}) \mod \Lambda_{B,k},
\]

(31)

where the inter-stream interference \( v_k \) is \( a-priori \) known when encoding the \( k \)-th stream at the BS (cf. (27b)).

The signal \( s_{M,k} \) at the \( k \)-th MS is encoded as

\[
s_{M,k} = \frac{1}{\alpha_k} ((c_{M,k} - d_{M,k}) \mod \Lambda_{M,k}),
\]

(32)

where \( d_{M,k} \) is a random dithering vector uniformly distributed over the Voronoi region of \( \Lambda_{M,k} \).

It is worth mentioning that the authors in [21] presumed that random dithering cannot be used in the first phase of the MIMO eTwRC due to the non-cooperation among MSs. Interestingly, we will show that through a careful precoder design, random dithering can be in fact employed to improve performance.

D. Relay’s Operation: Lattice Decoding

We now consider the operations at the relay. The relay’s decoding order is from the \( K \)-th stream to the first stream. For the \( k \)-th subchannel in (26), the relay intends to decode the combinations \( s_{B,k} + \alpha_k s_{M,k} + v_k \). Recall that \( \tilde{w}_k^{(k)} \) in (26) is a weighted sum of the network-coded signals already decoded; hence it can be cancelled from the received signal before Lattice decoding. Together with the knowledge of the dithering vectors \( d_{B,k} \) and \( d_{M,k} \), the relay constructs

\[
w_{R,k} = y_{R,k} + d_{B,k} + d_{M,k}
\]

(33a)

\[
= s_{B,k} + d_{B,k} + v_k + \alpha_k s_{M,k} + d_{M,k} + \psi_{R,k}
\]

(33b)

\[
= c_{B,k} + \tilde{c}_{B,k} + \tilde{\psi}_{R,k},
\]

(33c)

\(^1\)It is always assumed that the dithering signals, such as \( d_{X,k} \), are globally known to all the nodes in the network.
where \( \tilde{c}_{B,k} = s_{B,k} + d_{B,k} + v_k \), and \( \tilde{c}_{M,k} = \alpha_k s_{M,k} + d_{M,k} \). From (31), we see that \( \tilde{c}_{B,k} = (c_{B,k} - v_k - d_{B,k}) \) is a lattice point in \( \Lambda_B \). Similarly, \( \tilde{c}_{M,k} = (c_{M,k} - d_{M,k}) \) is a lattice point in \( \Lambda_C \). Hence, \( (\tilde{c}_{B,k} + \tilde{c}_{M,k}) \in \Lambda_C \) is decodable using lattice decoding with a vanishing error probability, provided that [8]

From (30), we also obtain

\[
R_{B,k} \leq R_{B \rightarrow R,k}(P_{B,k}) = \left[ \frac{1}{2} \log \left( \frac{[R_B(k,k)]^2 P_{B,k}}{\sigma^2} \right) \right]^+. \tag{34}
\]

From (31), we see that

\[
R_{M,k} \leq R_{M \rightarrow R,k}(P_{M,k}) = \left[ \frac{1}{2} \log \left( \frac{[R_M(k,k)]^2 P_{M,k}}{\sigma^2} \right) \right]^+. \tag{35}
\]

E. Relay’s Operation: Re-encoding

After lattice decoding, the relay calculates

\[
s_{R,k} = (\tilde{c}_{B,k} + \tilde{c}_{M,k}) \mod \Lambda_B, \tag{36}
\]

for \( k = 1, \ldots, K \). Then, one-side DPC encoding is applied to \( s_{R,k} \), with a random order \( \mu_1, \ldots, \mu_K \), so that each MS receives an interference-free signal. Note that the decoding order \( \mu = [\mu_1, \ldots, \mu_K]^T \) is specified by \( \phi \) as \( \phi(k, \mu_k) = 1 \), and \( \phi(k, j) = 0 \) for \( j \not= \mu_k, k = 1, \ldots, K \). From (12), the received signal at the \( k \)-th MS can be expressed as

\[
y_{M,k} = l_{RM}(q_k, q_k) x_{R,DP C,q_k} + t_{M,k} + \tilde{\psi}_{M,k}, \tag{37}
\]

where \( x_{R,DP C,q_k} \) denotes the row of \( \psi_M \), \( t_{M,k} = \sum_{n=1}^{q_k-1} l_{RM}(q_n, q_k) x_{R,DP C,n} \) and \( x_{R,DP C,n} \) denotes the \( n \)-th DPC encoded signal. The interference \( t_{M,k} \) can be pre-cancelled at the BS using dirty paper precoding as

\[
x_{R,DP C,q_k} = \left( s_{R,k} - \beta_{M,k} \frac{t_{M,k}}{l_{RM}(q_k, q_k)} - d_{R,k} \right) \mod \Lambda_B, \tag{38}
\]

where \( d_{R,k} \) is a random dither vector that is known by the BS and the MSs, and \( \beta_{M,k} = \frac{[l_{RM}(q_k, q_k)]^2 P_{R,k}}{[l_{RM}(q_k, q_k)]^2 P_{R,k} + \sigma^2} \) is the minimum mean square error coefficient for decoding at the MS [25], with \( P_{R,k} \) being the transmit power of the \( k \)-th stream of the RS satisfying \( \sum_{k=1}^{K} P_{R,k} \leq P_R \). The DPC encoded signal \( x_{R,DP C} = [x_{R,DP C,1}, \ldots, x_{R,DP C,K}]^T \) is then linearly precoded as (11) and broadcast to the BS and MSs in the second phase.

F. Phase 2: MS Decoding

The decoding operation at each MS \( k \) is as follows: For \( k = 1, \ldots, K \), the \( k \)-th MS first decodes \( s_{R,k} \) from the received signal \( y_{M,k} \); then it recovers \( c_{B,k} \) from \( s_{R,k} \) (i.e., the decoded \( s_{R,k} \)) with the help of the knowledge of the self-message \( c_{M,k} \) and the dither signal \( d_{M,k} \). For the first step, it has been shown in [21] that the probability \( \hat{s}_{R,k} \neq s_{R,k} \) vanishes as \( T \rightarrow \infty \) provided that

\[
R_{B,k} \leq R_{B \rightarrow R,k}(P_{B,k}) = \left[ \frac{1}{2} \log \left( 1 + \frac{[l_{RM}(q_k, q_k)]^2 P_{R,k}}{\sigma^2} \right) \right]. \tag{39}
\]

Here we focus on the second step, i.e., to recover \( c_{B,k} \) from \( s_{R,k} \). Note that \( s_{R,k} \) in (40) can be written as

\[
s_{R,k} = (\tilde{c}_{B,k} + \tilde{c}_{M,k}) \mod \Lambda_B = (s_{B,k} + d_{B,k} + v_k + \alpha_k s_{M,k} + d_{M,k}) \mod \Lambda_B = (c_{B,k} + (c_{M,k} - d_{M,k}) \mod \Lambda_B, \quad \text{mod } \Lambda_B, \tag{40}
\]

where the second equality follows from

\[
(x \mod \Lambda_{M,k}) \mod \Lambda_{B,k} = x \mod \Lambda_{B,k} \]

As \( c_{M,k} \) and \( d_{M,k} \) are known, the \( k \)-th MS obtains \( c_{B,k} \) as

\[
c_{B,k} = (\hat{s}_{R,k} - (c_{M,k} - d_{M,k}) \mod \Lambda_{M,k} - d_{M,k}) \mod \Lambda_B = (\hat{s}_{R,k} - s_{R,k} + c_{B,k}) \mod \Lambda_B = c_{B,k}, \tag{41}
\]

G. Phase 2: BS Decoding

The BS first decodes \( s_{R,k}, k = 1, \ldots, K \), from the received signal \( \tilde{Y}_B \) in (14). Note that inter-stream interference still exists at the BS since DPC encoding is only applied to the RS-MS link. However, if the decoding order at the BS is the same as the encoding order at the RS, it was shown in [21] that this interference can be successively cancelled at the BS since \( I_{RB} \) is lower-triangular. The corresponding probability of \( \hat{s}_{R,k} \neq s_{R,k} \) vanishes as \( T \rightarrow \infty \), provided that

\[
R_{M,k} \leq R_{R \rightarrow B,k}(P_{R,k}) = \left[ \frac{1}{2} \log \left( 1 + \frac{[l_{RB}(q_k, q_k)]^2 P_{R,k}}{\sigma^2} \right) \right]. \tag{42}
\]

With the knowledge of the self-message \( c_{B,k} \), the BS then recovers \( c_{M,k} \) from \( \hat{s}_{R,k} \) by calculating

\[
\hat{c}_{M,k} = (\hat{s}_{R,k} - c_{B,k}) \mod \Lambda_{M,k} \tag{43}
\]

To see this, we obtain from (40) that

\[
\hat{c}_{M,k} = (\hat{s}_{R,k} - s_{R,k} + (c_{M,k} - d_{M,k}) \mod \Lambda_{M,k} + d_{M,k}) \mod \Lambda_{M,k} = (\hat{s}_{R,k} - s_{R,k} + c_{M,k}) \mod \Lambda_{M,k} = c_{M,k}, \tag{44}
\]

where the last step holds provided \( \hat{s}_{R,k} = s_{R,k} \).
H. Achievable Rates of the Overall Scheme

Combining the discussions in Subsections C-G, we have the following theorem for the proposed two-way relaying scheme.

**Theorem 1:** As $T \rightarrow + \infty$, a rate tuple of $(R_{B,1}, \ldots, R_{B,K}, R_{M,1}, \ldots, R_{M,K})$ of the MIMO cTWRC is achievable if

$$R_{B,k} \leq \min \left( R_{B \rightarrow R,k}(P_{B,k}), R_{R \rightarrow M,k}(P_{R,k}) \right),$$

$$R_{M,k} \leq \min \left( R_{M \rightarrow R,k}(P_{M,k}), R_{R \rightarrow B,k}(P_{R,k}) \right),$$

$$k = 1, \ldots, K,$$ where $R_{B \rightarrow R,k}(P_{B,k}), R_{R \rightarrow M,k}(P_{R,k}), R_{M \rightarrow R,k}(P_{M,k})$ and $R_{R \rightarrow B,k}(P_{R,k})$ are given in (34), (38), (39) and (41), respectively.

IV. ANALYSIS OF THE SUM-RATE PERFORMANCE

In this section, we analyze the sum-rate performance of the proposed two-way relaying scheme in the high SNR regime. We first consider the cut-set bound in (5). It is known that, in (35) and (41), respectively.

**Theorem 2:** The proposed scheme achieves the cut-set bound (45) under certain conditions. We next show that the achievable sum-rate of the BS-to-MS link is no greater than the relayed RS-MS link if the transmit power of the BS satisfies

$$P_B \leq \rho_B P_R,$$ where $\rho_B = \frac{k}{\prod_{k=1}^{K} (\lambda_{RM,k}/\lambda_{BR,k})}$. On the other hand, for the proposed scheme with equal power allocation, the transmission rate of the $k$-th spatial stream from the BS to the RS is less than or equal to that from the RS to the $k$-th MS if

$$P_B \leq \rho_{B,k} P_R,$$ where $\rho_{B,k} = |l_{RM}(q_k, q_k)|^2 / |l_{BR}(k, k)|^2$. We will show that if both (46) and (47) hold for all the $K$ spatial streams, i.e., $P_B \leq \min (\rho_B, \rho_{B,k}) P_R$, then the proposed scheme achieves the sum-rate cut-set bound of the BS-to-MS link in the high SNR regime. Similarly, the proposed scheme asymptotically achieves the sum-rate capacity if the data transmission of the BS-to-MS link is bottle-necked by the RS-MS link. Furthermore, it can be shown that the cut-set bound of the MS-to-BS link can be achieved in the high SNR regime if the data transmission from the MSs to the BS is bottle-necked by either the MS-RS link or the RS-BS link. Following these lines, we define the conditions C1-C4 as follows:

C1: $P_B \leq \min (\rho_B, \min_k \rho_{B,k}) P_R$;

C2: $P_B \geq \max \left( \rho_B, \max_k \rho_{B,k} \right) P_R$;

C3: $P_R \leq \min \left( \rho_M \prod_{k=1}^{K} P_{M,k}^{1/K}, \rho_{B,k} \right)$;

C4: $P_R \geq \max \left( \rho_M \prod_{k=1}^{K} P_{M,k}^{1/K}, \rho_{B,k} \right)$,

where $\rho_M = \lambda_{M,R}(k, k)/\lambda_{BR,R}(k, k)$, and $\rho_{B,k} = |l_{RM,k}(k, k)|^2 / |l_{BR,k}(q_k, q_k)|^2$. Then we can establish that:

**Theorem 2:** The proposed scheme achieves the cut-set bound (45) if one of the conditions C1-C2 holds and one of the conditions C3-C4 holds for a given DPC order $\Phi$.

**Proof:** See Appendix A.

Note that the conditions C1-C4 depend on the DPC order at the BS (i.e. different encoding order $\Phi$ results in different conditions). Theorem 2 states that the proposed scheme is able to asymptotically achieve the sum capacity of the MIMO cTWRC, provided that there exists such a DPC order that the data exchanges between the BS and the MSs are simultaneously constrained by either the BS-RS link or the MS-BS link. In contrast, for the precoding and decoding scheme proposed by Yang in [21], it is required that the SNR from the BS to the RS is much higher than the SNR from the RS to the MSs, such that the power inefficiency incurred by the precoding operation at the BS is negligible. Hence, the Yang’s scheme in [21] can only asymptotically achieve the sum capacity when the data exchanges are simultaneously bottlenecked by the MS-BS link. For the proposed novel precoding/decoding design, only unitary transforms are involved for the channels, and interference-free PNC can be performed with QR decomposition based SIC at the relay. As a result, the asymmetry between the BS-MS and MS-BS links is no longer a prerequisite to achieve near-capacity performance for the proposed scheme.

V. OPTIMAL POWER ALLOCATION

We have shown that, under certain conditions, the proposed two-way relaying scheme achieves the sum-capacity of the
MIMO cTWRC with equal power allocation in the high SNR regime. To fully exploit the potential of the proposed scheme, we next investigate the optimal power allocation at the BS and the RS to maximize the weighted sum-rate of the MIMO cTWRC at general SNR (which includes the sum-rate in \[45\] as a special case where all weights are equal). We will show that this optimization problem can be formulated as a monotonic program and can be solved by a polyblock outer approximation algorithm.

It is clear that, the optimal power allocation for each MS is to transmit with maximum power \( P_{M,k} \). Let \( P_B := [P_{B,1}, \ldots, P_{B,K}]^T \) and \( P_R := [P_{R,1}, \ldots, P_{R,K}]^T \) be the power allocation profiles at the BS and the RS, respectively, and \( P := [P_B^T, P_R^T]^T \). Consider the following weighted sum-rate maximization problem:

\[
\max_{P} R_{ws} = \sum_{k=1}^{K} (\xi_{B,k} R_{B,k} + \xi_{M,k} R_{M,k}) \tag{48a}
\]

s.t.

\[
\sum_{k=1}^{K} P_{B,k} \leq P_B, \quad \sum_{k=1}^{K} P_{R,k} \leq P_R, \tag{48b}
\]

where \( \xi_{B,k} \) and \( \xi_{M,k} \) denote the weights assigned to the \( k \)-th data stream at the BS and the data stream of the \( k \)-th MS, respectively.

The problem \[48\] is not convex. Yet, it can be reformulated as a monotonic program, which could be efficiently solved by a polyblock outer approximation method \[23\]. Using \([\log(x)]^+ = \log(1 + (x - 1)^+)\), the achievable rate \( R_{B,k} \) in \[44\] can be expressed as

\[
R_{B,k} = \frac{1}{2} \log(1 + SNR_{BM,k}(P)), \tag{49}
\]

where

\[
SNR_{BM,k}(P) = \min \left( \frac{|r_{BR}(k,k)|^2 P_{B,k}}{\sigma^2} - 1, \left| r_{RM}(q_k,q_k) / \sigma^2 \right| P_{R,k} \right)^+. \tag{50}
\]

Similarly, \( R_{M,k} \) in \[44\] can be expressed as

\[
R_{M,k} = \frac{1}{2} \log(1 + SNR_{MB,k}(P)), \tag{51}
\]

where

\[
SNR_{MB,k}(P) = \min \left( \frac{|r_{MR}(k,k)|^2 P_{M,k}}{\sigma^2} - 1, \left| r_{RB}(q_k,q_k) / \sigma^2 \right| P_{R,k} \right)^+. \tag{52}
\]

Then, the weighted sum-rate \( R_{ws} \) in \[48\] can be expressed as

\[
R_{ws} = \sum_{i=1}^{2K} \frac{\xi_i}{2} \log(1 + SNR_i(P)). \tag{53}
\]

Define the set \( S := \{P \mid \sum_{k=1}^{K} P_{B,k} \leq P_B, \sum_{k=1}^{K} P_{R,k} \leq P_R, k = 1, \ldots, K\} \). Introducing an auxiliary vector \( z = [z_1, \ldots, z_{2K}]^T \), we can rewrite \[48\] as

\[
\max_{z} \Gamma(z) := \sum_{i=1}^{2K} \frac{\xi_i}{2} \log(z_i). \tag{54}
\]

where the feasible set \( Z := \{z \mid 1 \leq z_i \leq 1 + SNR_i(P), i = 1, \ldots, 2K, \forall P \in S\} \).

\[
G := \{z \mid 0 \leq z_i \leq 1 + SNR_i(P), i = 1, \ldots, 2K, \forall P \in S\}. \tag{55}
\]

It can be shown that \( G \) is a compact normal set with nonempty interior \[23\]. Further \( H := \{z \mid z_i \geq 1, \forall i\} \) is a reverse normal set. Then, \[54\] becomes a standard MP \[23\] as

\[
\max_{z} \Gamma(z) \quad \text{s. t. } z \in G \cap H. \tag{56}
\]

For the MP in \[56\], we can use a polyblock outer approximation method to find its global optimal solution \[23\]. This method has been used for power control \[28\], multicell coordinated beamforming \[29\], etc. The main idea of the iterative polyblock outer approximation algorithm is to construct a series of outer polyblocks \( P_n \) to approximate \( G \cap H \). Given any finite set \( T_n = \{v_i \mid i = 1, \ldots, I\} \), the union of all the sets \( \{x \mid 0 \leq x \leq v_i\} \) is a polyblock with vertex set \( T_n \). A polyblock \( P_n \) is an outer polyblock of \( S \) if \( S \subseteq P_n \). Usually, the algorithm starts from a one-vertex outer polyblock \( P_0 \) of \( G \cap H \), and a smaller new outer polyblock is constructed in each iteration. A key step in the construction of the new outer polyblock \( P_{n+1} \) from \( P_n \) is to find the following projection \[23\]:

\[
\theta^n = \max \{\alpha \mid \alpha z^n \in G\}, \tag{57}
\]

where \( z^n = \arg \max_{x \in T_n} \Gamma(z) \) denotes the maximizer among the vertices in \( T_n \). With \( \theta^n \), the projection \( y^n = \theta^n z^n \) is then the unique point where the halfline from \( 0 \) through \( z^n \) meets the upperbounding of \( G \). From \[55\], \( \theta^n \) can be determined as

\[
\theta^n = \max \{\alpha \mid \alpha z^n \in G\} = \max \{\alpha \mid \alpha \leq \min_{i=1,\ldots,2K} \frac{1 + SNR_i(P)}{z^n_i}, \forall P \in S\} \tag{58}
\]

With the definitions of \( SNR_{BM,k}(P) \) in \[50\] and \( SNR_{MB,k}(P) \) in \[52\], the above max-min problem \[58\] can be decoupled into the following two sub-problems:

**P1:** \( \theta^P_i = \max_{P_{B,i}} \min_{P_{R,i}} \frac{1}{z^n_k} \left( 1 + \left( \frac{|r_{BR}(k,k)|^2 P_{B,k}}{\sigma^2} - 1 \right)^+ \right) \)

s.t. \( \sum_{k=1}^{K} P_{B,k} \leq P_B \).
Lemma 1: \[ \min(\theta) \]

Note that the optimal \( \theta^* \) to find an \( \epsilon \)-max-min solution to (58), we propose the following algorithm. \( \ell = z_{n+k} \) for (58) is then given by

\[ \min_{k=1, \ldots, K} \left( z_{K+k} \left( 1 + \frac{|l_{RM}(q, q_k)|^2 P_{M,k}}{\sigma^2} \right) \right) \]

s.t. \( \sum_{k=1}^K P_{R,k} \leq P_R \).

The optimal \( \theta^* \) for (58) is then given by \( \theta^* = \min(\theta_1^*, \theta_2^*, \theta_3^*) \), where

\[ \theta_3^* = \min_{k=1, \ldots, K} \left( \frac{1}{z_{n+k}} \left( \frac{1}{z_{n+k}} \sum_{k=1}^K \frac{P_B}{\sigma_n^2 \sigma_{n+1}^2} \right)^{\frac{1}{2}} \right), \]

where \( \ell \) is an integer satisfying \( 1 \leq \ell \leq K \), and \( \frac{1}{z_{n+1}} \sum_{k=1}^K \frac{z_{n+k} \sigma_{n+1}^2}{\sigma_n^2} \leq P_B < \frac{1}{z_{n+1}} \sum_{k=1}^K \frac{z_{n+k} \sigma_{n+1}^2}{\sigma_n^2} \). For P2, the optimal \( \theta_2^* \) is given by solving the following equation:

\[ \sum_{k=1}^K \left[ \max \left( \frac{(\theta_2^* z_{n+k} - \sigma_2^2)}{|l_{RM}(q, q_k)|^2}, \frac{(\theta_2^* z_{n+k} - \sigma_2^2)}{|l_{RB}(q, q_k)|^2} \right) \right] \]

Proof: see Appendix B.

Let \( z^{opt} \) denote the global optimal solution of (58). For a given accuracy tolerance level \( \epsilon > 0 \), we say that a feasible \( \bar{z} \) is an \( \epsilon \)-optimal solution if \((1 + \epsilon) \Gamma(\bar{z}) \geq \Gamma(z^{opt})\). Based on the max-min solution to (58), we propose the following algorithm to find an \( \epsilon \)-optimal solution to (58).

Algorithm 1: for weighted sum-rate maximization

**Initialization:** select an accuracy level \( \epsilon > 0 \), let \( n = 0 \), and CurrentBestValue(CBV) = \(-\infty\). Initialize vertex set \( T_0 \) with a selected outer vertex to construct the initial outer polyblock \( P_0 \).

**Repeat:**

1. Find \( \bar{z}^n \in T_n \) that maximizes \( \Gamma(z) \) and solve (58) to obtain \( \theta^* \) and \( \theta_n = \theta^* z^n \).
2. If \( y^n \in \mathcal{H} \) and \( \Gamma(y^n) > \text{CBV} \), then \( \text{CBV} = \Gamma(y^n) \) and \( \bar{z} = y^n \).
3. Let \( z^n(i) = z^n - (z_i^n - y^n_i)e_i, i = 1, \ldots, 2K \), where \( z_i^n \) and \( y_i^n \) are the \( i \)-th entry of \( z^n \) and \( y^n \), respectively; i.e., \( z^n(i) \) is obtained by simply replacing the \( i \)-th entry of \( z^n \) by \( y_i^n \). Clearly, \( y_i^n \leq z_i^n(i) \leq z_i^n \).
4. Let \( T_{n+1} = (T_n \setminus \{z^n\}) \cup \{z^n(i)\} \cap \mathcal{H} \); i.e., obtain a new vertex set \( T_{n+1} \) by replacing the vertex \( z^n \) in \( T_n \) with \( 2K \) new vertices \( z^n(i), i = 1, \ldots, 2K \). Further remove from \( T_{n+1} \) any \( v_j \in T_{n+1} \) that satisfying \( \Gamma(v_j) \leq \text{CBV}(1 + \epsilon) \). By construction of \( z^n(i) \), the new outer polyblock \( P_{n+1} \) with vertex set \( T_{n+1} \) satisfies \( P_{n+1} \subseteq P_n \).
5. Set \( n = n + 1 \) and goto Step 1 until \( T_n \) is empty.

**Output:** \( \bar{z} \) and CBV as the \( \epsilon \)-optimal solution for (58).

Per iteration of Algorithm 1, we have \( y^n = \theta^n z^n \in \mathcal{G} \). If \( y^n \in \mathcal{H} \) too, we obtain a feasible point \( y^n \in \mathcal{G} \cap \mathcal{H} \). In this case, we update \( \text{CBV} = \max \{ \text{CBV}, \Gamma(y^n) \} \). This implies \( \text{CBV} \) is the current best value so far, and the corresponding \( \bar{z} \) is the current best solution for (58). Observe that for any \( v_j \in T_{n+1} \) satisfying \( \Gamma(v_j) \leq \text{CBV}(1 + \epsilon) \), we have \((1 + \epsilon) \text{CBV} \geq \Gamma(y), \forall y \in [0, v_j] \), due to monotonicity of \( \Gamma(\cdot) \). Hence, \( v_j \) can be removed from \( T_{n+1} \) for further consideration since \( \bar{z} \) will be the desired \( \epsilon \)-optimal solution if \( z^{opt} \in [0, v_j] \).

Algorithm 1 is essentially a “smarter” branch-and-bound method. A key requirement for its guaranteed convergence is that \( z \in \mathcal{G} \cap \mathcal{H} \) is lower bounded by a strictly positive vector. Since \( z \geq 1 > 0 \) in (58), it readily follows from [23, Theorem 1] that

**Proposition 1:** Algorithm 1 globally converges to an \( \epsilon \)-optimal solution for (58).

**Remark 1:** For each iteration of Algorithm 1, only a simple bisection search is involved (see Lemma 1). Furthermore, as shown in Theorem 2, equal power allocation is near optimal in many cases. Hence, we can use the equal power allocation solution to construct the initial outer polyblock \( P_0 \), which can speed up the convergence of Algorithm 1.

Before leaving this section, we emphasize that the sum-rate optimization in this section works for a fixed DPC order at the relay, i.e., the permutation matrix \( \Phi \) in (10) is fixed. The global optimum will be obtained by enumerating over all possible DPC orders.

VI. FURTHER DISCUSSIONS

**A. Extension to General Antenna Setups and Multiple Relays**

The proposed two-way relaying scheme can be readily extended to the MIMO cTWRC where each MS is equipped with multiple antennas. In this case, a straightforward approach is to treat each antenna at the MS as a virtual “user” (though the virtual users associated with a common MS share the power budget of this MS). Then, the proposed transceiver and relaying scheme developed in Section III directly applies. Furthermore, as antenna cooperation is physically allowed, we can introduce an extra linear precoder at each MS. The MSs’ linear precoders as well as the power allocations at the BS and the RS can be designed using iterative optimization methods, such as the one in [14]. Closed-form suboptimal linear precoders can be also used, such as SVD-based eigen beamforming [7]. With an appropriate precoding design, the system performance can be further enhanced.

The proposed scheme can be extended to a more general antenna setup in which the number of antennas at the BS and the RS may be unequal. In this case, the total number of data streams that can be supported by the two-way relaying network is given by \( N_s = 2 \min(N_B, N_R, \sum_{k=1}^{2K} M, k) \) with \( N_s/2 \) streams in each way, where \( M, k \) represents the number
of antennas at the k-th MS. When the number of antennas at the BS (or RS) is larger than \( N_s/2 \) (implying that this node has antenna redundancy in supporting \( N_s/2 \) independent data streams), an extra linear precoder can be applied at the BS (or RS) for beamforming. Again, iterative optimization methods could be used to find these linear precoders. The detailed precoding design for a general antenna setup will be an interesting direction for future research.

The proposed scheme can be also extended to the MIMO cTWRC with multiple distributed relay nodes \([30]\). One straightforward approach is to select the best relay for two-way communications based on channel conditions. Then the proposed scheme directly applies. The system performance can be further improved if we jointly design encoding/decoding schemes at multiple relays. However, this is out of the scope of the present work and will be a good direction to pursue in our future work.

B. Channel Estimation in the MIMO cTWRC

For the proposed scheme, global CSI is assumed available at the BS and the RS for signal encoding/decoding and power allocation. We make this assumption in order to explore the fundamental performance limit of the cTWRC, as in the existing works \([16]-[20]\). From the viewpoint of practical implementation, CSI acquisition in two-way relay networks is challenging. Generally, the relay is deployed at fixed position, and the channel between the RS and BS is slow-fading or quasi-static. This simplifies the CSI estimation at the BS. For the channels between the relay and the MSs, training based method can be employed to estimate the CSIs as in conventional cellular networks. These estimated CSIs at the RS can then be fed back to the BS. Considering the significant performance improvements brought by the proposed scheme and the fact that both BS and RS are powerful infrastructures, the complexity of the channel estimation in the MIMO cTWRC could be affordable.

VII. NUMERICAL RESULTS

In this section, numerical results are presented to demonstrate the performance of the proposed two-way relaying scheme. It is assumed that all elements of \( \mathbf{H}_{BR} \) and \( h_{k,R}, k = 1, \ldots, K, \) are independently drawn from a circularly symmetric complex Gaussian distribution with zero mean and unit variance. The channel matrices/vectors remain constant over the two phases and are reciprocal, i.e., \( \mathbf{H}_{BR} = \mathbf{H}_{BR}^H, h_{R,k} = h_{k,R}^T, \forall k \). All the MSs have the same power budget, i.e., \( P_{M,k} = P_M, \forall k \), and all the nodes have the same noise variance \( \sigma^2 \). The SNRs are defined as \( SNR_{X,R} = P_X / \sigma^2 \) and \( SNR_{R,X} = P_R / \sigma^2 \), where \( X \in \{B,M\} \). The SNR in the following figures is defined as \( SNR = SNR_{MR} = P_M / \sigma^2 \). Unless otherwise specified, it is also assumed that \( N_M = 1, N_B = N_R = K \), and each MS exchanges one data stream with the BS.

Fig. 2 shows the sum-rate of the proposed scheme for a cTWRC with \( K = 4 \) single-antenna MSs. Each node in the network has the same power budget, i.e., \( P_B = P_R = P_M \). Achieved sum-rate of the AF based interference alignment (AF-IA) scheme in \([17]\) and the scheme proposed by Yang in \([21]\) are included in Fig. 2 for comparison. For the proposed scheme, we also consider a low-complexity sub-optimal alternative in which the DPC encoding order at the RS is fixed as \( \Phi = I_K \) and equal power allocations are used for all spatial streams. A similar sub-optimal alternative for Yang’s scheme is also shown, in which the BS uses ZF-based precoding, the RS employs fixed-order DPC and the transmit powers are the same for all data streams. It is shown that AF-IA scheme in \([17]\) suffers from the noise propagation problem and the achievable sum-rate is much lower than the other two schemes. The proposed scheme with optimal permutation \( \Phi \) and power allocation can approach the cut-set bound as the SNR increase, the gap is only 0.5 bps/Hz when \( SNR \geq 25 \) dB. Compared with the sub-optimal scheme, there is about 1 bps/Hz gain for the optimal solution. It can be also seen that sum-rate gain is about 3 bps/Hz for the proposed scheme as compared to Yang’s scheme.

Fig. 3 compares the sum-rate performance of the proposed scheme and the Yang’s scheme \([21]\) when there are \( K = 8 \) and \( K = 16 \) single-antenna MSs in the cTWRC. Note that there are \( K! \) possible DPC encoding orders in total at the
RS. Even for a moderate $K$, it is too time-consuming to find the optimal DPC encoding order and the corresponding optimal power allocations. Instead, we search over $5 \times 10^4$ randomly chosen DPC encoding orders and use equal power allocation to determine the best DPC order. Even with this suboptimal approach, we can see from Fig. 3 that the proposed scheme performs only 0.5 dB away from the cut-set bound. Compared with Yang’s scheme, we see that the performance gain increases as the number of MSs increases. In particular, when there are 16 users in the network, more than 7 dB power gain is observed for the proposed scheme.

To further demonstrate the advantage of the proposed scheme, Fig. 4 illustrates the achievability of the cut-set bound under different $SNR_{BR}$. The transmit power of the RS is 10 dB higher than the MSs, and the SNRs are fixed to be $SNR_{MR} = 30$ dB, $SNR_{RM} = SNR_{RR} = 40$ dB. It can be seen that the proposed scheme is able to achieve the cut-set bound when $SNR_{BR} \geq 20$ dB. However, for Yang’s scheme, it is required that $SNR_{BR}$ is much larger than $SNR_{RM}$ to meet the condition for the rate-loss due to precoding at the BS to be negligible.

The weighted sum-rate performance of the proposed scheme is shown in Fig. 5. We assume that the priority of the data transmission from the BS to the MSs are higher than that from the MSs to the BS. The weights are chosen as $\xi_B,k = 0.4$ and $\xi_M,k = 0.1, \forall k$. Again, the proposed scheme is able to approach the weighted sum-rate cut-set bound when the SNR is higher than 25 dB.

Finally, Fig. 6 shows the performance of the MIMO cTWRC with $K = 2$ multi-antenna MSs. The number of antennas at the BS and RS is set to $N_B = N_R = 4$ and each MS is equipped with $N_M = 2$ antennas. Each MS exchanges two data streams with the BS; at any of the two MSs, no extra precoding is applied, i.e., each antenna transmit an independent data stream. Again, we see from Fig. 6 that the sum-rate upper bound of the MIMO cTWRC can be asymptotically achieved within 0.5 bps/Hz.

VIII. CONCLUSION

We proposed a novel two-way DF relaying scheme for the MIMO cTWRC. A non-linear lattice precoder was proposed at the BS to pre-compensate for the inter-stream interference, which enables efficient interference-free lattice decoding at the relay. The sufficient conditions for the achievability of the sum-rate cut-set bound were derived. The optimal power allocation for weighted sum-rate maximization was also obtained through monotonic programming. Numerical results demonstrated that the proposed scheme outperforms the existing alternatives and closely approaches the cut-set bound in the high SNR regime.

APPENDIX A: PROOF OF THEOREM 2

Consider the proposed scheme with equal power allocation, i.e., $P_{B,k} = P_B/K$ and $P_{R,k} = P_R/K, \forall k$. In the high SNR regime, the achievable rates $R_{B \rightarrow R,k}(P_{B,k})$ and
\[ R_{R \rightarrow M,k}(P_{R,k}) \text{ can be expressed as} \]
\[
R_{B \rightarrow R,k}(P_{B,k}) = \left[ \frac{1}{2} \log \left( \frac{|r_{BR}(k,k)|^2 P_{B,k}}{\sigma^2} \right) \right]^+ \simeq \frac{1}{2} \log \left( \frac{|r_{BR}(k,k)|^2 P_{B,k}}{\sigma^2} \right),
\]
and
\[
R_{R \rightarrow M,k}(P_{R,k}) \simeq \frac{1}{2} \log \left( \frac{|l_{RM}(q_k,q_k)|^2 P_{R,k}}{\sigma^2} \right).
\]

We first consider Condition C1. If the inequalities in C1 holds, i.e.,
\[
P_B \leq \frac{|l_{RM}(q_k,q_k)|^2}{|r_{BR}(k,k)|^2} P_R,
\]
then we have
\[
R_{B \rightarrow R,k}(P_{B,k}) = \frac{1}{2} \log \left( \frac{|r_{BR}(k,k)|^2 P_B}{K \sigma^2} \right) \leq R_{R \rightarrow M,k}(P_{R,k}),
\]
i.e., the achievable rate for the \( k \)-stream of the BS-RS link is lower than or equal to that from the RS to the \( k \)-th MS. Consequently, the sum-rate of the proposed scheme with equal power allocation for the BS-to-MS link is given by
\[
R_{\text{sum},B \rightarrow M} = \sum_{k=1}^{K} R_{B \rightarrow R,k}(P_{B,k}) = \sum_{k=1}^{K} \frac{1}{2} \log \left( \frac{|r_{BR}(k,k)|^2 P_B}{K \sigma^2} \right)
\]
\[
= \frac{1}{2} \log \prod_{k=1}^{K} \frac{|r_{BR}(k,k)|^2 P_B}{K \sigma^2}
\]
\[
= \frac{1}{2} \log \det \left( \frac{P_B}{K \sigma^2} H_{BR} H_{BR}^H \right)
\]
\[
= \sum_{k=1}^{K} \frac{1}{2} \log \left( \frac{\lambda_{BR,k} P_B}{K \sigma^2} \right).
\]

Now consider the cut-set bound in (45). Note that C1 implies \( P_B \leq \rho_B P_R \), then the sum-rate bound of the BS-to-MS link is given by
\[
R_{\text{sum,cs},B \rightarrow M} = \sum_{k=1}^{K} R_{B,k} = \sum_{k=1}^{K} \frac{1}{2} \log \left( \frac{\lambda_{BR,k} P_B}{K \sigma^2} \right).
\]

From (65) and (66), we see that the gap between the proposed scheme and the cut-set upper bound of the BS-to-MS link vanishes as \( \frac{P_B}{\sigma^2} \to +\infty \). Therefore, the proposed scheme asymptotically achieves the cut-set upper bound of the BS-to-MS link when condition C1 holds. Similarly, it can be shown that when C2 holds, the proposed scheme asymptotically achieves the cut-set upper bound of the MS-to-BS link as \( \frac{P_B}{\sigma^2} \to +\infty \).

Further, it can be shown in a similar way that the sum-rate gap between the proposed scheme and the cut-set bound for the MS-to-BS link vanishes as \( \frac{P_B}{\sigma^2} \to +\infty \) and \( \frac{P_B}{\sigma^2} \to +\infty \) when either C3 or C4 holds. This concludes the proof of Theorem 2.

### Appendix B: Proof of Lemma 1

For P1, it is clear that \( \sum_{k=1}^{K} P_{B,k} = P_B \) holds for the optimal \( \theta_1 \). Then P1 can be written as:
\[
\theta_1^n = \max_{P_B} \min_{k=1,\ldots,K} \frac{1 + \left( \frac{P_{B,k}}{\sigma^2} - 1 \right)}{z_{n,k}}
\]
\[
\text{s.t.} \sum_{k=1}^{K} P_{B,k} = P_B,
\]
where \( \sigma_k^2 = \sigma^2 / |r_{BR}(k,k)|^2 \).

Sort \( \{z_{n,k}\}^K_{k=1} \) that \( z_{n,1} \leq z_{n,2} \leq \ldots \leq z_{n,K} \) and define \( z_{n,0} = 0 \). Note that \( \theta_1^n \geq \frac{1}{z_{n,1}} \). Depend on the value of \( P_B \), there are two possible cases:

**Case A:** \( \theta_1^n = \frac{1}{z_{n,1}} \), where \( 1 \leq \ell \leq K \). The corresponding power allocation satisfies
\[
P_{B,\pi_k} = 0, \quad \frac{1 + \left( \frac{P_{B,\pi_k}}{\sigma_{\pi_k}^2} - 1 \right)}{z_{n,\pi_k}} = \frac{1}{z_{n,\pi_k}}, \quad k=1,\ldots,\ell-1,
\]
\[
0 \leq P_{B,\pi_\ell} \leq \sigma_{\pi_\ell}^2, \quad 1 + \left( \frac{P_{B,\pi_\ell}}{\sigma_{\pi_\ell}^2} - 1 \right) = \frac{1}{z_{n,\pi_\ell}},
\]
\[
P_{B,\pi_\ell} > \sigma_{\pi_\ell}^2, \quad 1 + \left( \frac{P_{B,\pi_k}}{\sigma_{\pi_k}^2} - 1 \right) = \frac{1}{z_{n,\pi_k}}, \quad k=\ell+1,\ldots,K.
\]

From (68a), we have \( P_{B,\pi_k} = \frac{z_{n,\pi_k}^2}{\sigma_{\pi_k}^2} \), for \( k = \ell + 1, \ldots, K \).

Then, \( P_B = \sum_{k=1}^{K} P_{B,k} = \sum_{k=\ell+1}^{K} P_{B,\pi_k} + P_{B,\pi_\ell} \), and we have
\[
\frac{1}{z_{n,\pi_\ell}} \sum_{k=\ell+1}^{K} z_{n,\pi_k}^2 \sigma_k^2 \leq P_B \leq \sum_{k=\ell+1}^{K} \frac{z_{n,\pi_k}^2}{z_{n,\pi_\ell}} \sigma_k^2.
\]

**Case B:** \( \frac{1}{z_{n,1}} < \theta_1^n < \frac{1}{z_{n,2}} \), where \( 1 \leq \ell \leq K \). The corresponding power allocation satisfies
\[
P_{B,\pi_k} = 0, \quad \frac{1 + \left( \frac{P_{B,\pi_k}}{\sigma_{\pi_k}^2} - 1 \right)}{z_{n,\pi_k}} = \frac{1}{z_{n,\pi_k}}, \quad k=1,\ldots,\ell-1
\]
\[
P_{B,\pi_\ell} > \sigma_{\pi_\ell}^2, \quad 1 + \left( \frac{P_{B,\pi_k}}{\sigma_{\pi_k}^2} - 1 \right) = \theta_1^n, \quad k=\ell+1,\ldots,K.
\]

From (70b), we have \( P_{B,\pi_k} = \frac{z_{n,\pi_k}^2}{\sigma_{\pi_k}^2} \theta_1^n \), for \( k = \ell + 1, \ldots, K \).

Then, \( P_B = \sum_{k=1}^{K} P_{B,k} = \sum_{k=\ell+1}^{K} P_{B,\pi_k} = \sum_{k=\ell+1}^{K} \frac{z_{n,\pi_k}^2}{\sigma_{\pi_k}^2} \theta_1^n \), and \( \theta_1^n = \frac{1}{z_{n,\pi_\ell}} \). From \( \frac{1}{z_{n,\pi_\ell}} < \theta_1^n < \frac{1}{z_{n,\pi_{k=1}}} \), we obtain the constraints on \( P_B \) as
\[
\frac{1}{z_{n,\pi_k}} \sum_{k=\ell}^{K} z_{n,\pi_k}^2 \sigma_k^2 < P_B < \frac{1}{z_{n,\pi_{k=1}}} \sum_{k=\ell}^{K} z_{n,\pi_k}^2 \sigma_k^2.
\]

Combining the above two cases we arrive at (59).
For P2, the optimal solution must satisfy

\[
\frac{1}{\gamma_k} \left( 1 + \frac{\|RM(q_k,k)\|^2 P_{R,k}}{\sigma^2} \right) \geq \theta_2^n,
\]

which implies that \( P_{R,k} \geq \frac{(\theta_2^n + 1 - 1)^2}{\|RM(q_k,k)\|^2} \). Similarly, we have

\[
P_{R,k} \geq \left( \frac{(\theta_2^n + 1 - 1)^2}{\|RM(q_k,k)\|^2} \right)^2. \]

On the other hand, \( P_{R,k} \geq 0 \), hence we have

\[
P_{R,k} \geq \left[ \max \left( \frac{(\theta_2^n + 1 - 1)^2}{\|RM(q_k,k)\|^2}, \frac{(\theta_2^n + 1 - 1)^2}{\|RB(q_k,k)\|^2} \right) \right]^+. \]

As a result, the maximum \( \theta_2^n \) can be determined by solving the equation (60) via a simple bisection search.

**References**


